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Some properties of the supersoluble formation and the supersoluble residual of a group

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Abstract

Purpose: In this paper, We determine the finite group $G = HK$ such that K is a supersoluble subgroup of G, and H is not a supersoluble subgroup of G.

Methods: Let p, q, r be primes such that $p < q < r$, and p, q are not a divisor of $r - 1$, and p is not a divisor of $q - 1$. Let X be a group of order p, and let $F = GF(q)$ and $L = GF(r)$ such that the filed F contains a primitive pth root of unity. Let V be a simple FX-module, and let $Y = V \rtimes X$ and W also be a faithful simple LY-module. Let $G = W \rtimes Y$, $H = W \rtimes X$, and $K = W \rtimes V$.

Results: Then, we determine that K is a supersoluble subgroup of G, and H is not a supersoluble subgroup of G.

Conclusions: We characterize the supersoluble residual of group G.

Keywords: Supersoluble, Formation, $\mathfrak{X}$-residual, Supersoluble residual, FX-module

Introduction

This paper continues a thread of research in finite soluble groups initiated by Ballester-Bolinhes et al. [1]. It is shown in [2] that a finite group G, which is the product of two normal supersoluble subgroups, is supersoluble if and only if G' is nilpotent. Asaad and Shaalan (Theorem 3.8 in [3]) proved the following generalization of Baer's result:

Assume that a finite group G is the product of the supersoluble subgroups H and K. Assume further that G' is nilpotent. If H commutes with every subgroup of K and K commutes with every subgroup of H, then G is supersoluble.

They also prove an analogous result by considering K nilpotent instead of G' (Theorem 3.2). Later, Carocca [4] presented extensions of the preceding result considering p-supersolubility instead of supersolubility. Following Carocca [4], we say that the subgroups H and K of a group G are mutually permutable if H commutes with every subgroup of K and K commutes with every subgroup of H. If $G = HK$ and H and K are mutually permutable, we say that G is the mutually permutable product of the subgroups H and K.

It is known that the class $\mathfrak{U}$ of all finite supersoluble groups is a formation. This means that if a finite group G is supersoluble and N is a normal subgroup of G, then G/N is supersoluble, and if M and N are two normal subgroups of a finite group G, then $G/(M \cap N)$ is supersoluble, provided that G/M and G/N are supersoluble. Consequently, every finite group G has a smallest normal subgroup with a supersoluble quotient. This subgroup is called the supersoluble residual of G, and it is denoted by $G^{\mathfrak{U}}$. It is clear that $G^{\mathfrak{U}}$ is epimorphism-invariant, and so, it is a characteristic subgroup of G (see Lemma 2.4, Chapter II in [5]).

This paper focuses on the study of supersoluble subgroups and the supersoluble residual of the group $G = [W][V]X$ as a semidirect product and considers the subgroups $H = W \rtimes X$ and $K = W \rtimes V$ of G such that X is the cyclic group of order p, and V is an irreducible and faithful X-module over $GF(q)$, and $Y = V \rtimes X$ is the corresponding semidirect product, and W is an irreducible and faithful Y-module over $GF(r)$ such that p, q and r are primes. We determine that G is the mutually permutable product of the subgroups H and K. Moreover, H is not a supersoluble subgroup of G. On the other hand, $K \in \mathfrak{U}$ and $H^{\mathfrak{U}} < W$. However, $G^{\mathfrak{U}} = W$.

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Methods

Preliminaries

Whenever possible, we follow the notation and terminology of [5,6]. All groups considered are finite.

Definition 2.1. *[4]. Let G be a group and H and K be subgroups of G. We say that H and K are mutually permutable if H commutes with every subgroup of K and K commutes with every subgroup of H.*

Definition 2.2. *[5]. A class of groups is a collection $\mathfrak{X}$ of groups with the property that if $G \in \mathfrak{X}$ and if $H \cong G$, then $H \in \mathfrak{X}$. We will often use the term $\mathfrak{X}$-group to describe a group belonging to $\mathfrak{X}$.*

Class $\mathfrak{U}$ denotes the class of finite supersoluble groups.

Definition 2.3. *[5]. If $\mathfrak{X}$ and $\mathfrak{I}$ are classes of groups, we define their class product $\mathfrak{X}\mathfrak{I}$ as follows:*

$\mathfrak{X}\mathfrak{I} = (G : G$ *has a normal subgroup* $N \in \mathfrak{X}$ *with* $G/N \in \mathfrak{I})$.

If $\mathfrak{X} = \emptyset$ or $\mathfrak{I} = \emptyset$, we have the obvious interpretation $\mathfrak{X}\mathfrak{I} = \emptyset$. For powers of a class, we set $\mathfrak{X}^0 = (1)$, and for $n \in \mathbb{N}$, make the inductive definition $\mathfrak{X}^n = (\mathfrak{X}^{n-1})\mathfrak{X}$.

Definition 2.4. *[5].*

(a) A class map $\mathtt{c}$ is called a closure operation if, for all classes $\mathfrak{X}$ and $\mathfrak{I}$, the following three conditions are satisfied:

Co1: $\mathfrak{X} \subseteq \mathtt{c}\mathfrak{X}$ (we say $\mathtt{c}$ is expanding);
Co2: $\mathtt{c}\mathfrak{X} = \mathtt{c}(\mathtt{c}\mathfrak{X})$ (we say $\mathtt{c}$ is idempotent);
Co3: If $\mathfrak{X} \subseteq \mathfrak{I}$, then $\mathtt{c}\mathfrak{X} \subseteq \mathtt{c}\mathfrak{I}$ (we say is monotonic).

(b) A class $\mathfrak{X}$ is said to be $\mathtt{c}$-closed if $\mathfrak{X} = \mathtt{c}\mathfrak{X}$. (If $\mathtt{c}$ is a closure operation, it is clear from Co2 that $\mathtt{c}\mathfrak{I}$ is $\mathtt{c}$-closed for any class $\mathfrak{I}$.) We adopt the convention that the empty class $\emptyset$ is $\mathtt{c}$-closed for every closure operation c.

(c) The product $\mathtt{AB}$ of two class maps is defined by composition; thus,

$$(\mathtt{AB})\,\mathfrak{X} = \mathtt{A}(\mathtt{B}\mathfrak{X})$$

for all classes $\mathfrak{X}$.

Definition 2.5. *[5]. For a class of groups, we define:*
$Q\mathfrak{X}$=(G: $\exists H \in \mathfrak{X}$ and an epimorphism from H onto G);
$R_0\mathfrak{X}$=(G: $\exists N_i \trianglelefteq G$(i=1,...,r) with $G/N_i \in \mathfrak{X}$ and $\bigcap_{i=1}^{r} N_i = 1$);
$E_\phi\mathfrak{X}$=(G: $\exists N \trianglelefteq G$ with $N \leq \Phi(G)$ and $G/N \in \mathfrak{X}$).

Definition 2.6. *[5]. A formation is a class of groups that is closed under both Q and R_0.*

Corollary 2.7. *Let $\mathfrak{X}$ be a class of groups, then $\mathfrak{X}$ is a formation if and only if the following two conditions are satisfied for the class $\mathfrak{X}$:*

(1) If $G \in \mathfrak{X}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{X}$.
(2) If N_1 and N_2 are normal subgroups of group G such that $G/N_1 \in \mathfrak{X}$ and $G/N_2 \in \mathfrak{X}$ and $N_1 \cap N_2 = 1$, then $G \in \mathfrak{X}$.

Proof. Straightforward. □

Definition 2.8. *[5]. An E_ϕ−closed classs is called saturated.*

Corollary 2.9. *Let $\mathfrak{X}$ be a formation. Then, $\mathfrak{X}$ is saturated if and only if for all finite groups G, $G/\Phi(G) \in \mathfrak{X}$ implies $G \in \mathfrak{X}$.*

Proof. Straightforward. □

Some properties of the supersoluble formation

We study in this section some properties of the supersoluble formation $\mathfrak{U}$. The next result includes the definition of the $\mathfrak{X}$-residual $G^{\mathfrak{X}}$ of a group G; it always exists if the class $\mathfrak{X}(\neq \emptyset)$ is R_0-closed, and it is epimorphism-invariant when $\mathfrak{X}$ is a formation.

Corollary 3.1. *The class $\mathfrak{U}$ is a saturated formation.*

Proof. By Huppert's Theorem [7], it is straightforward. □

Lemma 3.2. *(Lemma 2.4, Chapter II in [5]). Let $\mathfrak{X}$ be an R_0-closed class and G a finite group. Then the set $L = \{N \trianglelefteq G : G/N \in \mathfrak{X}\}$, partially ordered by inclusion, has a unique minimal element, denoted by $G^{\mathfrak{X}}$ and called the $\mathfrak{X}$-residual of G. It is a characteristic subgroup, and if $\mathfrak{X}$ is a formation and $\varepsilon : G \to \varepsilon(G)$ is an epimorphism, then $\varepsilon(G)^{\mathfrak{X}} = \varepsilon(G^{\mathfrak{X}})$.*

Corollary 3.3. *Let G be a finite group. Then,*

(1) If $H \trianglelefteq G$ and $G/H \in \mathfrak{U}$, then $G^{\mathfrak{U}} \leq H$;
(2) If $A \trianglelefteq G$ and $H \leq G$, then $(\frac{HA}{A})^{\mathfrak{U}} = \frac{H^{\mathfrak{U}}A}{A}$;
(3) If $H \leq G$, then $H^{\mathfrak{U}} \leq G^{\mathfrak{U}}$.

Proof. Straightforward. □

Lemma 3.4. *Let G be a finite group and H be a subgroup of G such that $(\frac{G}{A})^{\mathfrak{U}} = (\frac{HA}{A})^{\mathfrak{U}}$ where A is a normal subgroup of G. Then, $G^{\mathfrak{U}}A = H^{\mathfrak{U}}A$. Moreover, if $A \leq G^{\mathfrak{U}}$, then $G^{\mathfrak{U}} = H^{\mathfrak{U}}A$.*

Proof. By Corollary 3.3, $(\frac{HA}{A})^{\mathfrak{U}} = \frac{H^{\mathfrak{U}}A}{A}$. On the other hand, $(\frac{G}{A})^{\mathfrak{U}} = (\frac{GA}{A})^{\mathfrak{U}} = \frac{G^{\mathfrak{U}}A}{A}$. So, $\frac{G^{\mathfrak{U}}A}{A} = \frac{H^{\mathfrak{U}}A}{A}$, and then, $G^{\mathfrak{U}}A = H^{\mathfrak{U}}A$. If $A \leq G^{\mathfrak{U}}$, then $G^{\mathfrak{U}} = G^{\mathfrak{U}}A$. Therefore, $G^{\mathfrak{U}} = H^{\mathfrak{U}}A$. ☐

Proposition 3.5. *Let G be a finite group, A be a minimal normal subgroup of G, and H be a subgroup of G. If* $(\frac{G}{A})^{\mathfrak{U}} = (\frac{HA}{A})^{\mathfrak{U}}$, *then either* $A \leq G^{\mathfrak{U}}$ *or* $H^{\mathfrak{U}} = G^{\mathfrak{U}}$.

Proof. By Lemma 3.4, $G^{\mathfrak{U}}A = H^{\mathfrak{U}}A$. So $H^{\mathfrak{U}}(A \cap G^{\mathfrak{U}}) = H^{\mathfrak{U}}A \cap G^{\mathfrak{U}} = G^{\mathfrak{U}}A \cap G^{\mathfrak{U}} = G^{\mathfrak{U}}$; therefore, $H^{\mathfrak{U}}(A \cap G^{\mathfrak{U}}) = G^{\mathfrak{U}}$. On the other hand, $1 \leq A \cap G^{\mathfrak{U}} \leq A$ and $A \cap G^{\mathfrak{U}} \trianglelefteq G$. So, either $A \cap G^{\mathfrak{U}} = A$ or $A \cap G^{\mathfrak{U}} = 1$, and the proof is completed. ☐

The supersoluble residual of a group

All modules are right modules unless the contrary is stated.

Definition 4.1. *A module is said to be simple (irreducible) if*

(1) it is non-zero, and
(2) the only proper submodule that it possesses is the zero submodule.

An R-module M is called R-semisimple if M is a direct product of finitely many simple R-submodules.

Definition 4.2. *[8]. If G is a group and R is any ring with an identity element, the group ring RG is defined to be the set of all formal sums* $\sum_{x\in G} r_x x$ *where* $r_x \in R$ *and* $r_x = 0$ *with finitely many exceptions, together with the rules of addition and multiplication*

$$(\sum_x r_x x) + (\sum_x r'_x x) = \sum_x (r_x + r'_x)x;$$

and

$$(\sum_x r_x x)(\sum_x r'_x x) = \sum_x (\sum_{yz=x} r_y r'_z)x.$$

It is very simple to verify with these rules that RG is a ring with identity element $1_R 1_G$, which is simply written as 1.

Remark 4.3. *If F is a field, then FG, in addition to being a ring, has a natural F-module structure given by*

$$f(\sum_x f_x x) = \sum_x (ff_x)x, \qquad (f \in F).$$

Thus, FG is a vector space over F and $Dim_F(FG) = |G|$.

Definition 4.4. *The product of all the abelian minimal normal subgroups of a group G is called the abelian component of the socle and is denoted by* $Soc(G)$.

Theorem 4.5. *(Theorem 10.3, Chapter B in [5]). Let G be a finite group and K an arbitrary field. Then, the following conditions are equivalent:*

(a) G has a faithful simple module over K;
(b) $Soc_{\mathfrak{U}}(G)$ *has a subgroup N such that*
  (1) $Core_G(N)$*=1, and*
  (2) $Soc_{\mathfrak{U}}(G)/N$ *is cyclic and is a* p'*-group if* $char(K) = p > 0$.

Corollary 4.6. *Let p,q be primes and X be a group of order p. Let F be the Galois field* $F = FG(q)$. *Then, X has a faithful simple module over F.*

Lemma 4.7. *Let F be a field and X be a finite group. If V is a irreducible FX-module, then V is a vector space over F of finite dimension.*

Proof. Straightforward. ☐

Proposition 4.8. *(Lemma 9.2, Chapter B in [5]). Let G be an abelian group of order n, let K be a field, and let V be a simple KG-module. If either*

(1) the polynomial $x^n - 1$ *splits into a product of linear factors in K[x](in particular, if K contains a primitive nth root of unity), or*
(2) V is absolutely irreducible,

then $Dim_K(V) = 1$.

Corollary 4.9. *Let p,q be primes and X be a group of order p, let F=GF(q) and F contain a primitive pth root of unity. If V is a simple FX-module, then* $Dim_F(V)$*=1. Moreover,* $|V| = q$.

Lemma 4.10. *Let K be a field of prime characteristic p and G be a finite group. Let W be a KG-module. Then, W is an elementary abelian p-group.*

Proof. For every $w \in W$, $pw = \underbrace{w + \ldots + w}_{p\ times} = w(1_F 1_G) + \ldots + w(1_F 1_G) = w(\underbrace{1_F 1_G + \ldots 1_F 1_G}_{p\ times}) = w((1_F + \ldots + 1_F)1_G) = 0$. So, the abelian group $(W, +)$ is a p-elementary abelian group. ☐

Results and discussion

Theorem 5.1. *Let p,q,r be distinct primes such that* $p < q < r$. *Let X be a group of order p, and let* $F = GF(q)$ *and*

$\overline{K} = GF(r)$ such that the field F contains a primitive pth root of unity. Let V be a simple FX-module over F, and let $Y = V \rtimes_{\varphi} X$ such that for all $x \in X$ and for all $v \in V$, $v\varphi_x = v_x (= v(1_F x))$ where $\varphi_x \in Aut(V)$ and W also be a simple KY-module over $\overline{K}$. If $G = W \rtimes_{\psi} Y =$ such that for all $y \in Y$, $\psi(y) = \psi_y$, and for all $w \in W$, $w\psi_y = wy$ and $H = W \rtimes X$ and $K = W \rtimes V$. Then, G is the product of the mutually permutable subgroups H and K.

Proof. It is easy to verify that φ and ψ are well defined because, by Lemma 4.7, V is a vector space over F of finite dimension, and W is a vector space over $\overline{K}$ of finite dimension. By Lemma 4.10, $|W| = r^{\alpha}$ such that α is a nonnegative integer. On the other hand, by Corollary 4.9, $|V| = q$. Thus, $|G| = pqr^{\alpha}$ and $|H| = pr^{\alpha}$ and $|K| = qr^{\alpha}$. Therefore, $|G : K| = p$ and $K \trianglelefteq G$. Therefore, K commutes with every subgroup of H. Let x be an arbitrary element of X and w be an arbitrary element of W. Let $V =< v >$; then, □

$$\begin{aligned}(0,(v,o)) + (w,(0,x)) &= (w\psi_{-(v,0)}, (v,0) + (0,x))\\ &= (w\psi_{-(v,0)}, (v,x)).\end{aligned}$$

Now, let $t \in \mathbb{Z}$, $w' \in W$ and $x' \in X$. Then,

$$\begin{aligned}(w',(0,x')) + (0,(tv,0)) &= (w' + 0\psi_{-(0,x')}, (0,x') + (tv,0))\\ &= (w', ((tv)\varphi_{-x'}, x')).\end{aligned}$$

Let $x' = x$ and $w' = w\psi_{-(v,0)}$. There is a $t \in \mathbb{Z}$ such that $v\varphi_x = tv$, $(tv)\varphi_x^{-1} = v$; this means that $(tv)\varphi_{-x} = v$. Therefore,

$$\begin{aligned}&(0,(v,o)) + (w,(0,x)) = (w\psi_{-(v,0)}, (v,x))\\ &= (w', ((tv)\varphi_{-x}, x)) = (w',(0,x)) + (0,(tv,0))\\ &= (w\psi_{-(v,0)}, (0,x)) + (0,(tv,0))\\ &= (w\psi_{-(v,0)}, (0,x)) + t(0,(v,0)).\end{aligned}$$

Let $h \in H$ and $v_1 \in V$, then $v_1 = mv$ where $m \in \mathbb{Z}_q$, so $v_1 + h = mv + h$. Consequently, $v_1 + h = (m-1)v + v + h$; therefore, $v_1 + h = (m-1)v + h' + tv$ where $t \in \mathbb{Z}$ and $h' \in H$. There is a $s \in \mathbb{Z}$ such that $tv = s(mv)$, so $v_1 + h = (m-1)v + h' + s(mv)$. Therefore, $v_1 + h = h_1 + s'v_1$ where $h_1 \in H$ and $s' \in \mathbb{Z}$. Now, let $K_1 \leq K$ and $|K_1| = qr^{\beta}$ where $0 \leq \beta \leq \alpha$. We prove that H commutes with K_1. Let $W' = \{(w,(0,0)) | w \in W\}$ and $V' = \{(0,(v,0)) | v \in V\}$ and $X' = \{(0,(0,x)) | x \in X\}$. We know that $W' \trianglelefteq G$. Let $T \in Syl_r(K_1)$, then $n_r(K_1) = 1$. This means that $T \trianglelefteq K_1$. Let $S \in Syl_q(K_1)$, then $S \in Syl_q(K)$. Therefore, $S = V'^k$ where $k \in K$. On the other hand, $K = W' + V'$. So, $k = w_1 + v_1$ such that $v_1 \in V'$ and $w_1 \in W'$; therefore, $V'^k = -v_1 - w_1 + V' + w_1 + v_1$. We know that $w_1 + v_1 = v_1 + w_1'$ where $w_1' \in W'$, so $(V')^k = -(w_1 + v_1) + V' + (w_1 + v_1) = -(v_1 + w_1') + V' + (v_1 + w_1') = -w_1' - v_1 + V' + v_1 + w_1' = w_1' + V' + w_1 = (V')^{w_1'}$. Therefore, $S = (V')^{w_1'}$. $S \cap T = 1$, so $K_1 = S + T = T + S$. Let $h \in H, t \in T$, and $s \in S$, then T is a r-subgroup of G and $T \leq \mathcal{N}_G(H)$. Therefore, $(s+t) + h = s + (t+h) = s + (h' + t) = -w_1' + (w_1' + s - w_1') + w_1' + (h' + t)$, where $h' \in H$. Let $h_1 = w_1' + h_1'$ where $h_1 \in H$, then $(s+t) + h = -w_1' + (w_1' + s - w_1') + h_1 + t = -w_1' + h_1' + m(w_1' + s - w_1') + t$ where $h_1' \in H$ and $m \in \mathbb{Z}_q$. On the other hand, $m(w_1' + s - w_1') = w_1' + ms - w_1'$ and $ms - w_1' = w'' + ms$ where $w'' \in W'$. Therefore, $(s+t) + h = -w_1' + h_1' + w_1' + w'' + ms + t \in H + K_1$. This implies that $K_1 + H \subseteq H + K_1$. Consequently, $K_1 + H = H + K_1$; this means that H commutes with K_1. Let $L \leq K$ and $|L| = r^m$ such that $0 \leq m \leq \alpha$, then L is a r-subgroup of G and $L \leq W' \leq H \leq \mathcal{N}_G(H)$. Therefore, $L+H = H+L$. Now, let $L \leq K$ and $|L| = q$, then $L = (V')^k$ where $k \in K$. We know that $K = W' + V' (= V' + W')$, so let $k = x + w_1$ such that $x \in V'$ and $w_1 \in W'$. Therefore, $(V')^k = (V')^{x+w_1} = (V')^{w_1}$. Let $t \in L$ and $h \in H$, then $l + h = -w_1 + (w_1 + l - w_1) + w_1 + h = -w_1 + (w_1 + l - w_1) + h_1$, where $h_1 \in H$. Therefore, $l + h = -w_1 + h' + m(w_1 + l - w_1)$, where $h' \in H$ and $m \in \mathbb{Z}_q$. So, $l + h = -w_1 + h' + w - 1 + ml - w_1$. This yields $l + h = -w_1 + h' + w_1 + w_1' + ml$ where $w_1' \in W'$. Therefore, $l + h = -w_1 + h' + w_1 + w_1' + ml \in H + L$; this means that H commutes with L. Consequently, H commutes with every subgroup of K. Let $(w,(v,x)) \in G$, then $(w,(v,0)) + (0,(0,x)) = (w + 0\psi_{-(v,0)}, (v,0) + (0,x)) = (w,(v,x))$, where $(w,(v,0)) \in K$ and $(0,(0,x)) \in H$. This implies $G = H + K$, and the proof is completed.

Theorem 5.2. *Let the conditions of Theorem 5.1 be valid and p, q to be not a divisor of $r-1$ and p to be not a divisor of $q-1$. If the simple KY-module W will be faithful over K, then H is not a supersoluble subgroup of G.*

Proof. Let H be supersoluble. We also let $|W| = r^{\alpha}$ where α is a non-negative integer. If $|W| = 1$, then $Aut(W) = 1$; this means that $Y = ker\psi = 1$ (because W is a faithful simple KY-module over K), a contradiction. Let $|W| = r$, then $Aut(W) \cong \mathbb{Z}_{r-1}$. Therefore, $\frac{Y}{ker\psi} \hookrightarrow \mathbb{Z}_{r-1}$. This implies that $Y \hookrightarrow \mathbb{Z}_{r-1}$, a contradiction. Thus, $|W| = r^{\alpha}$ where $\alpha \geq 2$. If X is a maximal subgroup of H, then by Huppert's Theorem [7], $|H : X|$ is a prime, a contradiction. Therefore, X is not a maximal subgroup of H . Let M be a maximal subgroup H such that M contains X. Let $|H : M| = p_1$ where p_1 is a prime, and let $|M| = pk$ where $k \in \mathbb{Z}$, then $|H : M| = p_1$, so $p_1 | r^{\alpha}$. This implies that $p_1 = r$. So, $|M| = pr^{\alpha-1}$. Let $|Core_H(M)| = r^m p^n$ where $0 \leq n \leq 1$ and $0 \leq m \leq \alpha - 1$. On the other hand, $\frac{H}{Core_H(M)} \hookrightarrow S_{|H:M|}$ where $S_{|H:M|}$ is the symmetric group on $|H : M|$ letters.

Therefore, $r^{\alpha-m}p^{1-n}|r!$. If $\alpha-m \geq 2$, then $r^2|r^{\alpha-m}p^{1-n}|r!$, a contradiction. So, $\alpha - m = 1$; this means that $|Core_H(M)| = r^{\alpha-1}p^n$. If $n = 1$, then $M = Core_H(M)$. This yields $M \trianglelefteq H$. If $n = 0$, then $|Core_H(M)| = r^{\alpha-1}$. So, $|\frac{H}{Core_H(M)}| = pr$. On the other hand, $|\frac{M}{Core_H(M)}| = p$. Therefore, $\frac{M}{Core_H(M)} \in Syl_p(\frac{H}{Core_H(M)})$ and $n_p(\frac{H}{Core_H(M)})|r$, then $n_p(\frac{H}{Core_H(M)}) = 1$; this implies that $\frac{M}{Core_H(M)} \trianglelefteq \frac{H}{Core_H(M)}$. So, $M \trianglelefteq H$. Therefore, M is supersoluble. If $\alpha = 2$, then $|M| = pr$. We know that $X \in Syl_p(M)$ and $n_p(M) = 1$, then $X \trianglelefteq H$, a contradiction. So, $\alpha \geq 3$. X is not a maximal subgroup of M. Therefore, M has a maximal subgroup M_1 such that $X \leq M_1$. Similarly, we prove that s$|M_1| = pq^{\alpha-2}$ and $M_1 \trianglelefteq M$. Let $M_0(= M), ..., M_{\alpha-2}$ be subgroups of G such that $X \leq M_i$ and $M_i \trianglelefteq M_{i-1}$ and $|M_i| = pr^{\alpha-i-1}$, $(i = 1, ..., \alpha-2)$. So, $|M_{\alpha-2}| = pr$. Therefore, $n_p(M_{\alpha-2}) = 1$ and $X \leq M_{\alpha-2}$, then $X \in syl_p(M_{\alpha-2})$. This means that $XchM_{\alpha-2} \trianglelefteq M_{\alpha-3}$, so $X \trianglelefteq M_{\alpha-3}$. Inductively, we have $XchM \trianglelefteq H$. So, $X \trianglelefteq H$, a contradiction. Consequently, we imply that H is not supersoluble. □

Theorem 5.3. *Let p,q be primes such that $p < q$. Let G be a finite group and W,X be subgroups of G such that $G = WX$ and $|W| = q^{\alpha}$ ($\alpha \in \mathbb{N}$) and $|X| = p$. Also, let W be an abelian subgroup of G. If $[W, X] < W$, then $G^{\mathfrak{U}} < W$.*

Proof. Let $T = [W, X]$, so $T = [W, X] \trianglelefteq < W, X >= G$. Let $w \in W$ and $x \in T$; therefore, $[wT, xT] = T$. Thus, $[\frac{W}{T}, \frac{XT}{T}] = 1$, then $\frac{W}{T} \leq \mathcal{C}_{\frac{G}{T}}(\frac{XT}{T})$. If $|X \cap T| = p$, then $X \cap T = X$; this means that $X \leq T$, a contradiction. Consequently, $|X \cap T| = 1$. This yields $|\frac{XT}{T}| = p$, then $\frac{XT}{T}$ is abelian. So, $\frac{XT}{T} \leq \mathcal{C}_{\frac{G}{T}}(\frac{XT}{T})$. On the other hand, $n_q(G) = 1$. We have $\frac{G}{T} = \frac{W}{T}\frac{XT}{T} \leq \mathcal{C}_{\frac{G}{T}}(\frac{XT}{T})$; this yields $\frac{G}{T} = \mathcal{C}_{\frac{G}{T}}(\frac{XT}{T})$, so $\frac{XT}{T} \trianglelefteq \frac{G}{T}$. So, $\frac{G}{T} \in \mathfrak{U}$; therefore, $G^{\mathfrak{U}} \leq T < W$, and the proof is completed. □

Proposition 5.4. *Let p be a prime, $K = GF(p)$, H be a finite group, and W be an irreducible KH-module. Then, $G = W \rtimes_{\varphi} H$ is a group such that for all $h \in H$, $\varphi(h) = \varphi_h$ and for all $w \in W$ $w\varphi_h = wh (= w(1_K h))$, and W also is a minimal normal subgroup of G.*

Proof. It is easy to verify that the φ is well defined; this means that for every $h \in H, \varphi_h \in Aut(W)$. Thus, G is a group, and $W' = \{(w, 0)|w \in W\}$ is a normal subgroup of G. Let $T \trianglelefteq G$ and $T \leq W'$ and also $W_1 = \{w \in W|(w, 0) \in T\}$; this implies that $W_1 \leq W$. $G_1 = T + H_1 \leq G$ where $H_1 = \{(0, h)|h \in H\}$. Let $w \in W_1$ and $h \in H$. So, $(0, -h) + (w, 0) = (w\varphi_h, -h) \in G_1$, and this yields $(w\varphi_h, 0) \in T$, so $w\varphi_h \in W_1$. Let a be an arbitrary element of K. $w\underbrace{(1_K h + ... + 1_K h)}_{a\ times} = w(1_K h) + ... + (1_K h) = w\varphi_h + ... + w\varphi_h \in W_1$. So, $w(ah) \in W_1$. Now, let $w_1 \in W$ and $\lambda = \sum_{h \in H} a_h h \in KH$, then $w\lambda = \sum_{x \in H} w\lambda_x$ such that $\lambda_x = \sum_{h \in H} b_h^x h$ where $b_h^x = \begin{cases} a_x h = x \\ 0 h \neq x \end{cases}$. So, $w\lambda_x = w(a_x x)$; therefore, $\sum_{x \in h} w\lambda_x = \sum_{x \in H} w(a_x x) = w(\sum_{x \in h} a_x x) = w\lambda$. This means that for every $w \in W$ and $\lambda = \sum_{x \in H} a_x x, w\lambda = \sum_{x \in H} w(a_x x)$. Thus, for all $x \in H$ and for all $w \in W_1, w(a_x x) \in W_1$. So, $w\lambda \in W_1$; this means that W_1 is a KH-module. So, either $W_1 = 0$ or $W_1 = W$ because W is an irreducible KH-module and $W_1 \leq W$. Therefore, either $T = 0$ or $T = W'$; this implies that W' is a minimal normal subgroup of G. □

Theorem 5.5. *By hypothesis of Theorem 5.1, $G^{\mathfrak{U}} = W'$ such that $W' = \{(w, (0, 0))|w \in W\}$.*

Proof. We know that $|Y| = pq$, then if M is a maximal subgroup of Y, then either $|M| = p$ or $|M| = q$. By Huppert's Theorem [7], Y is supersoluble. On the other hand, $\frac{G}{W'} \cong Y$, so $\frac{G}{W'}$ is supersoluble, and then, $G^{\mathfrak{U}} \leq W'$. We know that $G^{\mathfrak{U}} \neq 1$(because by Theorem 5.2, H is not a supersoluble subgroup of G), by Proposition 5.4, W' is a minimal subgroup of G, then $G^{\mathfrak{U}} = w'$. □

Proposition 5.6. *[5]. Let V be a simple KG-module, let $N \trianglelefteq G$, and let W be a simple submodule of V_N. Then, the subset $W_g = \{wg|w \in W\}$ of V is a simple submodule of V_N, and $V = \bigoplus_{g \in G} W_g$. In particular, V_N is a semisimple KN-module.*

Proposition 5.7. *(Proposition 3.2 in [9]). Let M be an R-module. Then, the following statements are equivalent:*

(a) *M has a family $\{S_i\}_{i \in I}$ of simple submodules such that $M = \bigoplus_{i \in I} S_i (d.s)$;*
(b) *M has a family of simple submodules whose sum is M itself;*
(a) *every submodule of M is a direct summand of M.*

Theorem 5.8. *Let the hypothesis of Theorem 5.1 be valid. Then, $K \in \mathfrak{U}$.*

Proof. We know that $|K| = r^{\alpha}q$ where $\alpha \in \mathbb{N}$. If $\alpha = 1$, then by Huppert's Theorem [7], $K \in \mathfrak{U}$. Let $\alpha \geq 2$ and W_1 be a simple $\overline{K}V$-module of W_V where W_V is a semisimple $\overline{K}V$-module. By Proposition 4.8, $Dim_K(W_1) = 1$. Therefore, $|W_1| = r$. By Clifford's Theorem [5,10], $W = \bigoplus_{y \in Y} W_1 y$ such that for all $y \in Y$, $W_1 y$ is a simple $\overline{K}V$-module of W_V. By Proposition 5.7, $W = \bigoplus_{i \in I} W_1 y_i (d.s)$

where $\{y_i|i \in I\} \subseteq Y$. So $|W| = |\bigoplus_{i\in I} W_1 y_i| = r^{|I|}$. If $|W| = r^{\alpha}$, then $|I| = \alpha$. Therefore, W_V has a KV-module W' such that $|W'| = r^{\alpha-1}$. Now, let M be a maximal subgroup of K such that $|M| = q^{\gamma} r^{\beta}$ where $0 \leq \beta \leq \alpha$ and $0 \leq \gamma \leq 1$. If $\gamma = 0$, then $|M| = r^{\beta}$. We know that $n_r(K) = 1$ and M is a r-subgroup of K, then $M \leq W_I = \{(w,(0,0))|w \in W\}$. Therefore, $M = W_I$. Consequently, $|K : M| = q$. If $\gamma = 1$, then $|M| = r^{\beta} q$. Let $\beta = 0$, then $|M| = q$, so $M = V_I^k$ where $V_I = \{(0,(v,0))|v \in V\}$ and $k \in K$. Therefore, V_I is a maximal subgroup of K. Let $W_1' = \{(w,(0,0))|w \in W_1\}$, then $G_1 = W_1' + V_I$ is a subgroup of K. Consequently, $V_I \leq G_1$, a contradiction. Therefore, $\beta \geq 1$. Let $W'' \in Syl_r(M)$ and $V_1 \in Syl_g(M)$, then $M = W'' + V_1$. This implies that $M = W'' + (V_I)^k$ where $k \in K$. We know that $n_r(K) = 1$, then $W'' \leq W_I$; on the other hand, $M^{-k} = (W'')^{-k} + V_I$. Let $S = \{w|(w,(0,0)) \in (W'')^{-k}\}$. Let $w \in S$ and $v \in V$. $(0,(-v,0)) + (w,(0,0)) = (w\psi_{(v,0)},(-v,0) + (0,0)) = (w\psi_{(v,0)},(-v,0)) \in M^{-k}$. Therefore, there are $w_1 \in S$ and $v_1 \in V$ such that $(w\psi_{(v,0)},(-v,0)) = (w_1,(0,0)) + (0,(v_1,0))$. On the other hand, $(w_1,(0,0)) + (0,(v,0)) = (w_1,(0,0) + (v_1,0)) = (w_1,(v_1,0))$. So, $w\psi_{(v,0)} = w_1 \in S$. Consequently, there exists $i \in I$ such that $(W_1 y_i)' = \{(w,(0,0))|w \in W_1 y_i\} \nleq (W'')^{-k}$. Since, if for every $i \in I$, $\{(w,(0,0)|w \in W_1 y_i\} \leq (W'')^{-k}$ then $W_I \leq (W'')^{-k}$. Therefore, $(w'')^{-k} = W_I$. This yields $\beta = \alpha$, and this means that $M^{-k} = K$, a contradiction. Since $(W_1 y_i)' \nleq (W')^{-k}$, this implies that $(W_1 y_i)' \cap (W'')^{-k} = 1$. So, $|(W_1 y_i)' + (W'')^{-k}| = r^{\beta+1}$. Let $G' = ((W_1 y_i)' + (W'')^{-k}) + V_I$; the $G'$is a subgroup of K. We know that $|G'| = r^{\beta+1} q, M^{-k} \leq G'$ and M^{-k} is a maximal subgroup of K. Consequently, $G' = K$ and $\beta + 1 = \alpha$. So, $|M^{-k}| = r^{\alpha-1} q$ and $|K : M| = r$. By Huppert's Theorem [7], K is supersoluble, and the proof is completed. □

Conclusions

All our previous results show that the subgroup K of the finite group $G = HK$ is a supersoluble subgroup of G, and the subgroup H is not a supersoluble subgroup of G. Let p, q, r be primes such that $p < q < r$, and p, q are not a divisor of $r - 1$, and p is not a divisor of $q - 1$. Let X be a group of order p, and let $F = GF(q)$ and $L = GF(r)$ such that the filed F contains a primitive pth root of unity. Let V be a simple FX-module, and let $Y = V \rtimes X$ and W also be a faithful simple LY-module. Let $G = W \rtimes Y, H = W \rtimes X$, and $K = W \rtimes V$. Then, we determine that K is a supersoluble subgroup of G, and H is not a supersoluble subgroup of G, and we also characterize the supersoluble residual of group G.

Competing interests
The author declares that he has no competing interests.

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